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Harmonic analysis for differential forms on complex hyperbolic spaces

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Abstract

We use representation theory for the semisimple Lie group $G = SU(n, 1)$ to develop the L^2 harmonic analysis for differential forms on the complex hyperbolic space $H^n(\mathbb{C})$. In this setting, most of the basic notions and results known for functions are generalized: the abstract Plancherel theorem, the spectrum of the Hodge–de Rham Laplacian, the spherical function theory, the spherical Fourier transform and the Fourier transform. In addition, we calculate explicitly the Plancherel measure and estimate the decay at infinity of the heat kernel $H_t(e)$.

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1. Introduction

While harmonic analysis of *functions* on noncompact Riemannian symmetric spaces G/K is well understood for more than a quarter of a century (see the major reference [18]), harmonic analysis of *sections of homogeneous vector bundles* over the same spaces has been an intensive subject of study during essentially the last decade. Among many references, let us cite for instance [30,38,40] for the Poisson transform theory, [9,19,29,30,32] for the spherical function theory, and [8,9] for the Fourier transform theory. Most of these works deal with a quite large class of situations. On the other hand, restricting the study

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to particular bundles and/or to particular symmetric spaces allows more precise and more readable results: see e.g. [10,13,14,36,37].

The present article is devoted to the harmonic analysis on the bundle of differential forms over the n-dimensional complex hyperbolic space $G/K = SU(n,1)/S(U(n) \times U(1)) = H^n(\mathbb{C})$, which constitutes one of the three classical families of Riemannian symmetric spaces of noncompact type and of rank 1. The case of differential forms over real hyperbolic spaces was studied extensively in [31] (see [32,33] for an exposition), and here we use similar methods to develop the Fourier transform theory (inversion and Plancherel formulas with both algebraic and analytic points of view) as well as related geometric aspects (L^2 spectrum of the Hodge–de Rham Laplacian and behaviour of the heat kernel at the origin). As a matter of fact, part of the information was scattered throughout the literature and it is surprising that it was not put together — as far as we know — to answer natural questions of interest. Comments and historical references will be given throughout the paper.

The article is organized as follows: in Section 2 we recall some definitions and basic facts about differential forms on the Hermitian symmetric space $H^n(\mathbb{C})$. The main observation is that any differential form on $H^n(\mathbb{C}) = G/K$ can be viewed as a *function of (right) type τ on G*, where τ is a specific (complex) finite-dimensional unitary representation of K. More precisely, f is a function on G, having values in the representation space V_τ of τ, and such that

$$f(xk) = \tau(k)^{-1} f(x)$$

for any $x \in G$ and any $k \in K$. This allows in particular an identification between the space $L^2 \wedge^r H^n(\mathbb{C})$ of square-integrable differential r-forms on $H^n(\mathbb{C})$ and the space $L^2(G, K, \tau_r)$ of square-integrable functions of type τ_r on G, where τ_r is the rth exterior product of the complexified coadjoint representation of K on the dual $\mathfrak{p}^*$ of the tangent space $\mathfrak{p} \simeq \mathfrak{g}/\mathfrak{k} \simeq \mathbb{C}^n$ of G/K at the origin. The decomposition of τ_r into K-irreducible components corresponds then to the well-known *Lefschetz decomposition* into *primitive* elements of an r-form on the complex manifold $H^n(\mathbb{C})$.

In Section 3, by particularizing Harish-Chandra's Plancherel theorem, we state an abstract Plancherel formula for the space $L^2(G, K, \tau_r)$, i.e., we give its decomposition into G-irreducible components, which are either principal series or discrete series representations of the group G. The continuous part of the formula is carried out by examining the decomposition of the restriction of τ_r to the classical subgroup M of K, while the discrete part uses a general result due to Borel.

The calculations made in Section 3 are used in Section 4 to determine explicitly the L^2 spectrum of the Hodge–de Rham Laplacian. We construct also Poisson transforms (of Rumin differential forms) that are eigenforms for the Laplacian.

In Section 5, we give the analytic counterpart of the abstract Plancherel theorem. We develop first a *spherical Fourier transform* theory based on the study of *τ-spherical functions*. The main result is the inversion formula for this transform. Then, by standard arguments, we derive the inversion and Plancherel formulas for the *Fourier transform* of differential forms.

In Section 6 are computed very explicitly the Plancherel measures associated with the principal series representations that appeared in the Plancherel formulas previously stated. These expressions are used also in Section 7, which is devoted to the behaviour of the trace of the heat kernel $H_t(x)$ associated with differential forms on the complex hyperbolic space, when $x = e$ is the neutral element of $G = SU(n, 1)$ and $t \to +\infty$.

2. Notations and preliminaries

The *n-dimensional complex hyperbolic space* is the manifold

$$H^n(\mathbb{C}) := \{x \in \mathbb{C}^{n+1} : L(x, x) < 0\}/\mathbb{C}^*,$$

where L is the Hermitian Lorentz form

$$L(x, y) = \overline{y_1}x_1 + \cdots + \overline{y_n}x_n - \overline{y_{n+1}}x_{n+1}.$$

For our purpose, we shall view the complex hyperbolic space as the rank 1 symmetric space of noncompact type G/K, where $G = SU(n, 1)$ and $K = S(U(n) \times U(1))$ is the maximal compact subgroup of G which stabilizes the base point $o := (0, \dots, 0, 1)\mathbb{C}^*$.

Denote, respectively, by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of the groups G and K. As usual, write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition of $\mathfrak{g}$. The tangent space $T_o(G/K) \simeq \mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$ of $G/K = H^n(\mathbb{C})$ at the origin $o = eK$ will often be identified with the vector space $\mathbb{C}^n$ by means of the isomorphism

$$\begin{aligned} \mathfrak{p} &\longrightarrow \mathbb{C}^n, \\ X = \begin{pmatrix} 0_n & x \\ x^* & 0 \end{pmatrix} &\longmapsto x \end{aligned} \tag{2.1}$$

(with $x^* := {}^t\overline{x}$) so that the Euclidean inner product

$$g_o(X, Y) := \mathrm{Re}(y^* x) \tag{2.2}$$

on $\mathfrak{p}$ induces by translations a G-invariant Riemannian metric g on $H^n(\mathbb{C})$, for which this manifold has sectional curvature $-4 \le \kappa \le -1$. Similarly, the Hermitian inner product

$$h_o(X, Y) := y^* x \tag{2.3}$$

on $\mathfrak{p}$ induces a G-invariant Hermitian metric h on $H^n(\mathbb{C})$, for which $H^n(\mathbb{C})$ is a Kählerian manifold of dimension $2n$ over $\mathbb{R}$.

We recall now some basic facts about differential forms on the Hermitian symmetric space $H^n(\mathbb{C})$ (see e.g. [39, Ch. I and V] and [4, Ch. II]).

Let $0 \le r \le 2n$. *A (complex–valued) differential r-form on* $H^n(\mathbb{C})$ is a section of the vector bundle $\wedge^r E_{\mathbb{C}}$, where $E = T^* H^n(\mathbb{C})$ is the cotangent bundle over $H^n(\mathbb{C})$ (here and thereafter, the functors $\cdot^*$ and $\cdot_{\mathbb{C}}$, denote, respectively, algebraic duality and complexification of real vector spaces). We set

$$\Gamma \wedge^r H^n(\mathbb{C}) := \{\text{differential } r\text{-forms on } H^n(\mathbb{C})\},$$

$$\Gamma \wedge H^n(\mathbb{C}) := \bigoplus_{r=0}^{2n} \Gamma \wedge^r H^n(\mathbb{C}),$$

and shall replace 'Γ' by 'C_c', 'C^∞', 'C_c^∞' or 'L^2' if the sections encountered are, respectively, continuous with compact support, smooth, smooth with compact support or square integrable. The complex structure J on $\mathfrak{p}$ is both induced by the identification (2.1) and by the adjoint action of the generator

$$Z_0 = \begin{pmatrix} \dfrac{\mathrm{i}}{n+1} I_{n-1} & 0 \\ 0 & -\dfrac{n\mathrm{i}}{n+1} \end{pmatrix}$$

of the centre of $\mathfrak{k}$. Obviously, J extends to a $\mathbb{C}$-linear automorphism of $\mathfrak{p}_{\mathbb{C}}$ and, since $J^2 = -\mathrm{id}$, we denote by $\mathfrak{p}_\pm$ the eigenspace corresponding to the eigenvalue $\pm\mathrm{i}$, so that $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_- \simeq \mathfrak{p} \oplus \overline{\mathfrak{p}}$. Note that both metrics g and h are J-invariant. For $0 \le p, q \le n$, set

$$\begin{aligned} \wedge^{p,q}\mathfrak{p} &:= \wedge^p \mathfrak{p}_+ \otimes \wedge^q \mathfrak{p}_- \\ &= \mathrm{Vect}_{\mathbb{C}}\{u \wedge v,\ u \in \wedge^p \mathfrak{p}_+,\ v \in \wedge^q \mathfrak{p}_-\}. \end{aligned}$$

A (complex-valued) differential (p, q)-form on $H^n(\mathbb{C})$ is a section of the vector bundle $\wedge^{p,q} E$, and we denote by $\Gamma \wedge^{p,q} H^n(\mathbb{C})$ the set of such elements. The decomposition

$$\wedge^r \mathfrak{p}_{\mathbb{C}}^* = \wedge^r(\mathfrak{p}_+^* \oplus \mathfrak{p}_-^*) = \bigoplus_{p+q=r} \wedge^p \mathfrak{p}_+^* \otimes \wedge^q \mathfrak{p}_-^* = \bigoplus_{p+q=r} \wedge^{p,q}\mathfrak{p}^*$$

induces a similar decomposition on each fibre $\wedge^r E_{x,\mathbb{C}}$ over $x \in H^n(\mathbb{C})$, so that

$$\Gamma \wedge^r H^n(\mathbb{C}) = \bigoplus_{p+q=r} \Gamma \wedge^{p,q} H^n(\mathbb{C}). \tag{2.4}$$

Let us give now some essential identifications that will be used throughout this paper.

If τ is a unitary finite dimensional representation of K on a Hilbert space V_τ, we say that a function $f : G \to V_\tau$ is *of (right) type* τ if it verifies the relation

$$f(xk) = \tau(k)^{-1} f(x) \quad (\forall x \in G,\ \forall k \in K), \tag{2.5}$$

i.e., if f is a section of the homogeneous vector bundle $G \times_K V_\tau$ over G/K. We shall denote by $\Gamma(G, K, \tau)$ the space of functions of type τ on G and, as above, shall change 'Γ' for 'C_c', 'C^∞', 'C_c^∞' or 'L^2' when needed. Once we fix a Hermitian inner product on V_τ and a Haar measure $\mathrm{d}x$ on G, the scalar product on $L^2(G, K, \tau)$ is given by

$$(f_1, f_2) = \int_G \mathrm{d}x (f_1(x), f_2(x))_{V_\tau}. \tag{2.6}$$

Let Ad denote the adjoint representation of G. Since any element of K can be written as

$$k = \begin{pmatrix} U & 0 \\ 0 & v \end{pmatrix}, \quad U \in U(n),\ v \in U(1), \quad \text{with } \det U = v^{-1}, \tag{2.7}$$

K acts on $\mathfrak{p} \equiv \mathbb{C}^n$ by $\mathrm{Ad}(k)X \equiv Uxv^{-1}$, and this action preserves the complex structure J. Therefore

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_- \quad \text{and} \quad \mathfrak{p}_\pm^* \simeq \overline{\mathfrak{p}_\pm} \simeq \mathfrak{p}_\mp \tag{2.8}$$

as K-modules identities. For $0 \le r \le 2n$, let

$$\tau_r := \wedge^r(\mathrm{Ad}_+^* \oplus \mathrm{Ad}_-^*) \simeq \wedge^r(\mathrm{Ad}^* \oplus \overline{\mathrm{Ad}^*})$$

be the representation of K on $\wedge^r \mathfrak{p}_{\mathbb{C}}^* = \wedge^r(\mathfrak{p}_+^* \oplus \mathfrak{p}_-^*)$. Then it is well known that the bundle $\wedge^r E_{\mathbb{C}}$ can be viewed as $G \times_K V_{\tau_r}$, so that we have

$$\Gamma \wedge^r H^n(\mathbb{C}) \equiv \Gamma(G, K, \tau_r). \tag{2.9}$$

Similarly, for $0 \le p, q \le n$, $\wedge^{p,q} E \simeq G \times_K V_{\tau_{p,q}}$ and

$$\Gamma \wedge^{p,q} H^n(\mathbb{C}) \equiv \Gamma(G, K, \tau_{p,q}), \tag{2.10}$$

where

$$\tau_{p,q} := \wedge^p \mathrm{Ad}_+^* \otimes \wedge^q \mathrm{Ad}_-^* \simeq \wedge^p \overline{\mathrm{Ad}} \otimes \wedge^q \mathrm{Ad} \tag{2.11}$$

is the representation of K on $\wedge^{p,q}\mathfrak{p}^* = \wedge^p \mathfrak{p}_+^* \otimes \wedge^q \mathfrak{p}_-^*$. Thus

$$\tau_r = \bigoplus_{p+q=r} \tau_{p,q} \quad \text{and} \quad \Gamma(G, K, \tau_r) = \bigoplus_{p+q=r} \Gamma(G, K, \tau_{p,q}).$$

Remind also the classical isomorphisms

$$\begin{aligned} &\Gamma(G, K, \tau_{q,p}) \simeq \overline{\Gamma(G, K, \tau_{p,q})} \quad (\mathbb{C}\text{-conjugation isomorphism}),\\ &\Gamma(G, K, \tau_{p,q}) \simeq \Gamma(G, K, \tau_{n-q,n-p}) \quad (\text{Hodge duality}),\\ &\Gamma(G, K, \tau_r) \simeq \Gamma(G, K, \tau_{2n-r}) \quad (\text{idem}). \end{aligned}$$

Let $\Omega \in \Gamma(G, K, \tau_{1,1})$ be the (closed) fundamental form on $H^n(\mathbb{C})$ associated with the Hermitian metric h, i.e. for any $x \in H^n(\mathbb{C})$, $\Omega_x = \mathrm{Im}\, h_x \in \Gamma \wedge^{1,1} E_x$. Denote by $\omega \in \wedge^{1,1}\mathfrak{p}^*$ the corresponding element such that $\Omega_{gK} = g \times_K \omega$. Let L be the homogeneous operator of bidegree $(1,1)$ on $\wedge^{p,q}\mathfrak{p}^*$ which is left multiplication by ω. Remind that an element in $\wedge^{p,q}\mathfrak{p}^*$ is called *primitive* if it lies in the kernel of the L^2 adjoint L^* of L. We shall denote by $\tau'_{p,q}$ the restriction of $\tau_{p,q}$ to its primitive part, i.e., K acts on the subspace $\wedge_0^{p,q}\mathfrak{p}^*$ of primitive vectors in $\wedge^{p,q}\mathfrak{p}^*$ by $\tau'_{p,q}$. These representations are then irreducible and two by two inequivalent (use Lemma 6.2).

In our setting, the *Lefschetz decomposition* of an element of $\wedge^{p,q}\mathfrak{p}^*$ — or, analogously, of a (p,q)-form on $H^n(\mathbb{C})$ — (see [39, Ch. V, Theorem 3.12]) gives in the same time the K-decomposition of the representation $\tau_{p,q}$.

Proposition 2.1. *Let $0 \le p, q \le n$. Then*

$$\wedge^{p,q}\mathfrak{p}^* = \bigoplus_{k=0}^{\min(p,q)} L^k \wedge_0^{p-k,q-k} \mathfrak{p}^*$$

is a decomposition into irreducible K-modules. Moreover, for each k, $L^k \wedge_0^{p-k,q-k} \mathfrak{p}^*$ is K-isomorphic with $\wedge_0^{p-k,q-k}\mathfrak{p}^*$ and occurs with multiplicity 1.

Proof. See [20, Proposition 3.1] or [4, Ch. VI, Lemma 4.9]. □

In the sequel, we shall often refer to this result by writing more simply:

$$\tau_{p,q} = \bigoplus_{k=0}^{\min(p,q)} \tau'_{p-k,q-k}.$$

Since L and L^* extend naturally to G-invariant bundle operators, we have a corresponding decomposition for the space $\Gamma(G, K, \tau_{p,q})$. Moreover, since L and L^* commute with the Hodge–de Rham Laplacian Δ, it shows also that the spectrum of Δ is completely determined by the spectrum of its restriction to primitive differential forms (see Section 4).

Finally, let us point out that, by Hodge duality, we can, and shall from now on, restrict our study to $0 \le r \le n$, i.e., to $0 \le p+q \le n$.

3. The abstract Plancherel theorem

Since the Plancherel theorem (i.e. the decomposition into G-irreducible modules) for the space $L^2 \wedge^r H^n(\mathbb{C}) \equiv L^2(G, K, \tau_r)$ of square-integrable differential r-forms on $H^n(\mathbb{C})$ derives from the one of $L^2(SU(n,1))$, we recall first some standard facts of representation theory for the real rank one semisimple Lie group $G = SU(n,1)$. The material can be found in generality in, e.g., [23].

We begin with some notation. Let

$$H_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathfrak{p}. \tag{3.1}$$

Then $\mathfrak{a} := \mathbb{R}H_0$ is a Cartan subspace in $\mathfrak{p}$, and the corresponding analytic Lie subgroup A of G is parametrized by elements

$$a_t := \exp(tH_0) = \begin{pmatrix} \operatorname{ch} t & 0 & \operatorname{sh} t \\ 0 & I_{n-1} & 0 \\ \operatorname{sh} t & 0 & \operatorname{ch} t \end{pmatrix} \quad (t \in \mathbb{R}).$$

Let $\alpha \in \mathfrak{a}^*$ be defined by $\alpha(tH_0) = t$. Then $R(\mathfrak{g}, \mathfrak{a}) = \{\pm\alpha, \pm 2\alpha\}$ is a restricted root system of $(\mathfrak{g}, \mathfrak{a})$ with positive subsystem $R^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha, 2\alpha\}$ and corresponding Weyl group $W = W(\mathfrak{g}, \mathfrak{a}) \simeq \{\pm \mathrm{id}\}$. Later on, we shall often use the identification

$$\mathfrak{a}^*_{\mathbb{C}} \xrightarrow{\simeq} \mathbb{C},$$
$$\lambda\alpha \longmapsto \lambda.$$

Let $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be the sum of the positive root subspaces, N the corresponding analytic subgroup of G and ρ the half-sum of roots in $R^+(\mathfrak{g}, \mathfrak{a})$, counted with their multiplicities. Then $\rho = \frac{1}{2}(m_\alpha \alpha + m_{2\alpha} 2\alpha) = n\alpha$, since $m_\alpha = \dim \mathfrak{g}_\alpha = 2(n-1)$ and $m_{2\alpha} = \dim \mathfrak{g}_{2\alpha} = 1$. Remind the classical decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad G = KAN \text{ (Iwasawa)}, \quad G = K\{a_t, t \ge 0\}K \quad \text{(Cartan)}.$$

Let M be the centralizer of A in K and $P = MAN$ the usual (minimal) parabolic subgroup of G associated with A and N. Given $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \simeq \mathbb{C}$, the following action

$$(\sigma \otimes \mathrm{e}^{\mathrm{i}\lambda} \otimes \mathbf{1})(ma_t n) = \mathrm{e}^{\mathrm{i}\lambda t}\sigma(m)$$

defines a representation of P on the space V_σ. Then the principal series representation $\pi_{\sigma,\lambda} := \mathrm{Ind}_P^G(\sigma \otimes \mathrm{e}^{\mathrm{i}\lambda} \otimes \mathbf{1})$ of G acts on the space

$$\begin{aligned}\mathcal{H}_{\sigma,\lambda} &= L^2(G, MAN, \sigma \otimes \mathrm{e}^{\mathrm{i}\lambda} \otimes \mathbf{1}) \\ &:= \{f : G \to V_\sigma : f(xma_t n) = \mathrm{e}^{-(\mathrm{i}\lambda+\rho)t}\sigma(m)^{-1}f(x),\ f_{|K} \in L^2(K)\}\end{aligned}$$

by left translations: $\pi_{\sigma,\lambda}(g)f(h) = f(g^{-1}h)$. With this parametrization, $\pi_{\sigma,\lambda}$ is unitary if and only if λ is real. Moreover, unitary principal series representations are always irreducible, except maybe when $\lambda = 0$ (a criterion is recalled in the proof of Corollary 6.3). Note also that, as K-modules, $\mathcal{H}_{\sigma,\lambda}$ is isomorphic (for any λ) with the space $L^2(K, M, \sigma)$ of square integrable functions on K such that $f(km) = \sigma(m^{-1})f(k)$.

Since $\mathfrak{g}$ and $\mathfrak{k}$ have common rank n, the group G has *discrete series representations*, i.e., irreducible unitary representations with L^2 matrix coefficients. If $(\pi, \mathcal{H}_\pi)$ is such a representation, denote by d_π its formal degree. We let $\widehat{G}_d$ be the subset of discrete series in $\widehat{G}$.

For all real rank one semisimple connected Lie groups G, Harish-Chandra's Plancherel theorem can be stated as follows (see e.g. [25, Section 11]): for each $\sigma \in \widehat{M}$, there exists a Plancherel measure $\mathrm{d}\nu_\sigma(\lambda)$ on $\mathfrak{a}^*$ such that

$$L^2(G) \simeq \int_{W\backslash(\widehat{M}\times\mathfrak{a}^*)}^{\oplus} \mathrm{d}\nu_\sigma(\lambda)\mathcal{H}_{\sigma,\lambda}\widehat{\otimes}\mathcal{H}_{\sigma,\lambda}^* \oplus \bigoplus_{\pi\in\widehat{G}_d} \mathrm{d}_\pi\mathcal{H}_\pi\widehat{\otimes}\mathcal{H}_\pi^* \tag{3.2}$$

(the symbol $\widehat{\otimes}$ means Hilbert completion). The meaning of $W\backslash(\widehat{M} \times \mathfrak{a}^*)$ is the following: according to [5, Theorem 7.2], when λ is real, $\pi_{\sigma,\lambda}$ is unitarily equivalent with $\pi_{\sigma',\lambda'}$ if and only if there exists an element $w \in W$ such that the pair $(w \cdot \sigma, w \cdot \lambda)$ is equivalent with (σ', λ') (the action of such a w will be precised later on). Consequently, since the contribution of principal series must be taken up to unitary equivalence, one must reduce the support of the direct integral to only one representative of each equivalence class (σ, λ) of $\widehat{M} \times \mathfrak{a}^*$ under the action of W.

Now, let τ be a finite-dimensional unitary representation of K. Since $L^2(G, K, \tau)$ can be identified with $\{L^2(G) \otimes V_\tau\}^K$, where the upper index K stands for the space of K-invariant vectors for the right action of K on $L^2(G)$, (3.2) implies the decomposition

$$\begin{aligned}L^2(G, K, \tau) \simeq &\int_{W\backslash(\widehat{M}\times\mathfrak{a}^*)}^{\oplus} \mathrm{d}\nu_\sigma(\lambda)\mathcal{H}_{\sigma,\lambda}\,\widehat{\otimes}\,\mathrm{Hom}_K(\mathcal{H}_{\sigma,\lambda}, V_\tau) \\ &\oplus \bigoplus_{\pi\in\widehat{G}_d} \mathrm{d}_\pi\mathcal{H}_\pi\,\widehat{\otimes}\,\mathrm{Hom}_K(\mathcal{H}_\pi, V_\tau).\end{aligned} \tag{3.3}$$

Note that each vector space $\mathrm{Hom}_K(\cdot, V_\tau)$ is finite-dimensional, since every irreducible unitary representation of G is admissible. In the sequel, we shall denote by $L^2(G, K, \tau)_c$ (respectively by $L^2(G, K, \tau)_d$) the continuous (respectively discrete) part of $L^2(G, K, \tau)$ that corresponds to the decomposition (3.3). In order to reduce as much as possible this decomposition when $G = SU(n, 1)$, $K = S(U(n) \times U(1))$ and $\tau = \tau_r$, we must now determine whenever the spaces $\mathrm{Hom}_K(\mathcal{H}_{\sigma,\lambda}, V_{\tau_r})$ and $\mathrm{Hom}_K(\mathcal{H}_\pi, V_{\tau_r})$ are nontrivial.

3.1. Principal series representations decomposing $L^2(G, K, \tau_r)$

We remark first that, by Proposition 2.1, it suffices to determine $L^2(G, K, \tau)_c$ for $\tau = \tau'_{p,q}$, with $0 \le p + q \le n$. Since $\mathcal{H}_{\sigma,\lambda}$ is $L^2(K, M, \sigma)$ as a K-module, by Frobenius reciprocity,

$$\mathrm{Hom}_K(\mathcal{H}_{\sigma,\lambda}, V_{\tau'_{p,q}}) \simeq \mathrm{Hom}_M(V_\sigma, V_{\tau'_{p,q}}) \quad (\forall \lambda \in \mathfrak{a}^*_{\mathbb{C}}),$$

so that we must examine how the K-representations $\tau'_{p,q}$ restrict to the subgroup M. The method consists in calculating their highest weight, and then, in applying classical branching rules to get their M-decomposition.

With the choice of $\mathfrak{a}$ that was made previously, the Lie algebra $\mathfrak{m} \subset \mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(n-1) \times \mathfrak{u}(1))$ is constituted with elements

$$\begin{pmatrix} v & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & v \end{pmatrix}, \quad U \in \mathfrak{u}(n-1),\ v \in \mathfrak{u}(1), \quad \text{with } \mathrm{tr}\, U + 2v = 0. \tag{3.4}$$

Let $\mathfrak{h}$ (resp. $\mathfrak{t}$) denote the Cartan subalgebra of $\mathfrak{k}$ (resp. of $\mathfrak{m}$) constituted with diagonal elements. For $1 \le i \le n+1$, denote by ε_i the linear functional on $\mathfrak{h}_{\mathbb{C}}$ defined by $\varepsilon_i(\mathrm{diag}(h_1, \ldots, h_{n+1})) = h_i$. We shall keep the same notation for its restriction to $\mathfrak{t}_{\mathbb{C}}$. It is a classical result that the roots of the pairs $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ are, respectively,

$$R_K := R(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\varepsilon_i - \varepsilon_j,\ 1 \le i \ne j \le n\}, \tag{3.5}$$

$$R_M := R(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\varepsilon_i - \varepsilon_j,\ 2 \le i \ne j \le n\}, \tag{3.6}$$

while the corresponding positive subsystems (for the 'lexicographic' ordering) are given by

$$\begin{aligned} R_K^+ &= \{\varepsilon_i - \varepsilon_j,\ 1 \le i < j \le n\}, \\ R_M^+ &= \{\varepsilon_i - \varepsilon_j,\ 2 \le i < j \le n\}. \end{aligned} \tag{3.7}$$

Let $(e_i)_{i=1}^n$ denote the standard basis of $\mathfrak{p} \simeq \mathbb{C}^n$. Because of (2.7), the weights of the adjoint representation Ad of K are the $\varepsilon_i - \varepsilon_{n+1}$, with corresponding weight vectors e_i $(1 \le i \le n)$. Thus the highest weights of the irreducible representations $\wedge^p \overline{\mathrm{Ad}}$ and $\wedge^q \mathrm{Ad}$ of K are, respectively,

$$\mu_{\wedge^p \overline{\mathrm{Ad}}} = -\sum_{k=n-p+1}^{n} \varepsilon_k + p\,\varepsilon_{n+1}, \quad \mu_{\wedge^q \mathrm{Ad}} = \sum_{k=1}^{q} \varepsilon_k - q\,\varepsilon_{n+1}.$$

It follows then from definition (2.11) and from [4, Ch. VI, Lemma 4.9], that the highest weight of $\tau'_{p,q}$ is

$$\mu_{\tau'_{p,q}} = \sum_{k=1}^{q} \varepsilon_k - \sum_{k=n-p+1}^{n} \varepsilon_k + (p-q)\varepsilon_{n+1}. \tag{3.8}$$

Applying the branching laws from K to M (see [1, Theorem 4.4] or [2, Theorem 10.5]), we see that, generically,

$$\tau'_{p,q|M} = \sigma'_{p,q} \oplus \sigma'_{p-1,q} \oplus \sigma'_{p,q-1} \oplus \sigma'_{p-1,q-1}, \tag{3.9}$$

where each M-type $\sigma'_{a,b}$ occurs *with multiplicity* 1 and has highest weight

$$\mu_{\sigma'_{a,b}} = \frac{a-b}{2}(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{k=2}^{b+1} \varepsilon_k - \sum_{k=n-a+1}^{n} \varepsilon_k. \tag{3.10}$$

By 'generically', we mean that we set $\sigma'_{a,b} = 0$ if $\min(a,b) < 0$ or $\max(a,b) > n-1$, so that one or several factors may vanish in (3.9) for certain values of p and q.

Let us identify concretely the representations $\sigma'_{p,q}$. The space $\mathfrak{p}$ admits the decomposition $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2$, where

$$\mathfrak{q}_1 = \left\{ Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & y^* & 0 \end{pmatrix},\ y \in \mathbb{C}^{n-1} \right\},$$

$$\mathfrak{q}_2 = \left\{ Z = \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ -z & 0 & 0 \end{pmatrix},\ z \in \mathrm{i}\mathbb{R} \right\}.$$

With the analogue of notation (3.4) on the group level, the adjoint action of M on $\mathfrak{q}_1$ is given by $\mathrm{Ad}(m)Y \equiv Uyv^{-1}$. The complex vector space $\mathfrak{q}_1$ inherits from $\mathfrak{p}$ an $\mathrm{Ad}(M)$-invariant complex structure, so that we can write, similarly to (2.8), $\mathfrak{q}_1 = \mathfrak{q}_{1,+} \oplus \mathfrak{q}_{1,-}$ as an M-module decomposition. For $0 \le r \le 2(n-1)$ and $0 \le p, q \le n-1$, set

$$\sigma_r = \wedge^r(\mathrm{Ad}^*_+ \oplus \mathrm{Ad}^*_-),$$
$$\sigma_{p,q} = \wedge^p \mathrm{Ad}^*_+ \otimes \wedge^q \mathrm{Ad}^*_-,$$

so that $\sigma_r = \oplus_{p+q=r} \sigma_{p,q}$. Then it is easily checked, by computing weights, that the representation $\sigma'_{p,q}$ appearing in (3.9) is exactly the restriction of $\sigma_{p,q}$ to primitive vectors in $\wedge^p \mathfrak{q}^*_{1,+} \otimes \wedge^q \mathfrak{q}^*_{1,-}$.

We can now summarize our results and derive easy consequences.

Proposition 3.1. *Let $0 \le p+q \le n$. Then, with notations above:*

(I) *Set $\sigma'_{p,q} = 0$ if $\min(p,q) < 0$ or $\max(p,q) > n-1$. We have the following decomposition into irreducible inequivalent multiplicity 1 factors:*

$$\tau'_{p,q|M} = \sigma'_{p,q} \oplus \sigma'_{p-1,q} \oplus \sigma'_{p,q-1} \oplus \sigma'_{p-1,q-1}.$$

More precisely, seven disjoint cases must be distinguished:

(i) $\tau'_{p,q|M} = \sigma'_{p,q} \oplus \sigma'_{p-1,q} \oplus \sigma'_{p,q-1} \oplus \sigma'_{p-1,q-1}$ *if* $1 \le p, q \le n-2$ *and* $p+q \le n-1$;
(ii) $\tau'_{p,q|M} = \sigma'_{p-1,q} \oplus \sigma'_{p,q-1} \oplus \sigma'_{p-1,q-1}$ *if* $1 \le p, q \le n-1$ *and* $p+q = n$;
(iii) $\tau'_{0,q|M} = \sigma'_{0,q} \oplus \sigma'_{0,q-1}$ *if* $p = 0$ *and* $1 \le q \le n-1$;
(iv) $\tau'_{p,0|M} = \sigma'_{p,0} \oplus \sigma'_{p-1,0}$ *if* $q = 0$ *and* $1 \le p \le n-1$;
(v) $\tau'_{0,n|M} = \sigma'_{0,n-1}$ *if* $p = 0$ *and* $q = n$;
(vi) $\tau'_{n,0|M} = \sigma'_{n-1,0}$ *if* $p = n$ *and* $q = 0$;
(vii) $\tau'_{0,0|M} = \sigma'_{0,0}$ *if* $p = q = 0$.

(II) *We have the following decompositions into irreducible inequivalent factors:*

(i)
$$\tau_{p,q|M} = \sigma'_{p,q} \oplus \bigoplus_{k=1}^{\min(p,q)} (\sigma'_{p-k,q-k+1} \oplus \sigma'_{p-k+1,q-k}) \oplus \bigoplus_{k=1}^{\min(p,q)} 2\sigma'_{p-k,q-k};$$

(ii) *if* $0 \le r \le n-1$,

$$\tau_{r|M} = \bigoplus_{p+q=r} \sigma'_{p,q} \oplus \bigoplus_{p+q \le r-1} 2\sigma'_{p,q};$$

(iii) *if* $r = n$,

$$\tau_{n|M} = \bigoplus_{\substack{p+q \le n-1 \\ p+q \ne n-2}} 2\sigma'_{p,q} \oplus \bigoplus_{p+q=n-2} 3\sigma'_{p,q}.$$

Remark. *The decomposition* (3.9) *appeared previously in many references:* [4, Ch. VI, Section 4.10], [14, Section 1.2], [21, Section 4] and [6, Lemma 4.11].

If $\tau \in \widehat{K}$, denote by $\widehat{M}(\tau)$ the subset of $\widehat{M}$ whose elements occur in the decomposition of $\tau_{|M}$. As a consequence of the previous proposition, we see that each space $\mathrm{Hom}_K(\mathcal{H}_{\sigma,\lambda}, V_{\tau'_{p,q}})$ is one-dimensional, so that we can write:

$$L^2(G, K, \tau'_{p,q})_c = \int^{\oplus}_{W \backslash (\widehat{M}(\tau'_{p,q}) \times \mathfrak{a}^*)} d\nu_\sigma(\lambda) \mathcal{H}_{\sigma,\lambda}. \tag{3.11}$$

Thus, it remains only to examine the effect of the action of the Weyl group on the principal series representations. Remind that W can be realized as the quotient M'/M, where M' is the normalizer of A in K. Since $|W| = 2$, let us denote by $w = m'M$ the nontrivial element, where $m' = \begin{pmatrix} -I_2 & 0 \\ 0 & I_{n-1} \end{pmatrix}$. It acts on a representation $\pi_{\sigma,\lambda}$ by $w \cdot \pi_{\sigma,\lambda} = \pi_{w\cdot\sigma, w\cdot\lambda}$, where $w \cdot \sigma(m) = \sigma(m'^{-1} m m')$ and $w \cdot \lambda(H) = \lambda(\mathrm{Ad}(m')^{-1} H)$. But, since elements of $\mathfrak{t}_{\mathbb{C}}$ are diagonal, w acts trivially on the weights of any representation of $\widehat{M}$. In particular, for any $\sigma \in \widehat{M}(\tau'_{p,q})$, $w \cdot \sigma = \sigma$. On the other hand, it is clear that $w \cdot \lambda = -\lambda$. Hence $\pi_{\sigma,\lambda}$ is unitarily equivalent with $\pi_{\sigma,-\lambda}$. Consequently, the support in the integral in (3.11) can be reduced to $\widehat{M}(\tau'_{p,q}) \times \mathfrak{a}^*_+$, where $\mathfrak{a}^*_+$ denotes as usual the positive Weyl chamber in $\mathfrak{a}^*$ and can be identified with $\mathbb{R}^*_+ =]0, +\infty[$. Of course, as concerns the Plancherel measure $d\nu_\sigma(\lambda)$, the Weyl group invariance shows that is an even function of λ.

Our next result follows from these considerations and from Proposition 3.1.

Theorem 3.2. *Let $0 \leq r \leq n$. The continuous part of the Plancherel formula for the space $L^2(G, K, \tau_r)$ of L^2 differential r-forms on $H^n(\mathbb{C})$ is given by the following decompositions:*
(i) *if $0 \leq r \leq n-1$,*

$$L^2(G, K, \tau_r)_c = \bigoplus_{p+q=r} \int_{\mathbb{R}_+^*}^{\oplus} d\nu_{\sigma'_{p,q}}(\lambda)\, \mathcal{H}_{\sigma'_{p,q},\lambda} \oplus \bigoplus_{p+q \leq r-1} 2 \int_{\mathbb{R}_+^*}^{\oplus} d\nu_{\sigma'_{p,q}}(\lambda)\, \mathcal{H}_{\sigma'_{p,q},\lambda};$$

(ii) *if $r = n$,*

$$L^2(G, K, \tau_n)_c = \bigoplus_{\substack{p+q \leq n-1 \\ p+q \neq n-2}} 2 \int_{\mathbb{R}_+^*}^{\oplus} d\nu_{\sigma'_{p,q}}(\lambda) \mathcal{H}_{\sigma'_{p,q},\lambda} \oplus \bigoplus_{p+q=n-2} 3 \int_{\mathbb{R}_+^*}^{\oplus} d\nu_{\sigma'_{p,q}}(\lambda) \mathcal{H}_{\sigma'_{p,q},\lambda}.$$

Remark. *The Plancherel measure $d\nu_{\sigma'_{p,q}}(\lambda)$ will be explicitly determined in Section 6.*

3.2. Discrete series representations decomposing $L^2(G, K, \tau_r)$

We first recall a general result due to Borel ([3, Theorem A]) in the elementary presentation that was given in [31, Appendix A].

Let G/K be a Riemannian symmetric space of the noncompact type such that G and K have equal complex rank. Let $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ be a Cartan subalgebra, and denote by $R_K = R(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \subset R_G = R(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and by $W_K \subset W_G$ the corresponding root systems and Weyl groups. Once a positive subsystem R_K^+ in R_K is fixed, there are exactly $l = |W_G|/|W_K|$ positive subsystems in R_G whose intersection with R_K coincides with R_K^+. Let R_G^+ be one of them and let $i\mathfrak{h}_G^+$ denote the corresponding positive G-Weyl chamber in $i\mathfrak{h}$. Then any positive subsystem in R_G can be written as $w_j \cdot R_G^+$, where $w_1, \ldots, w_l$ are distinguished representatives of $W_K \backslash W_G$ in W_G. Let δ_G and δ_K denote, respectively, the half-sums of roots in R_G^+ and R_K^+. Recall that discrete series representations are, up to equivalence, uniquely determined by their Harish-Chandra parameter $w_j \cdot \Lambda$, where $\Lambda \in (i\mathfrak{h}_G^+)^*$ is such that $\Lambda + \delta_G$ is analytically integral (see e.g. [23, Section IX.7]). Finally, let τ_r be the representation of K such that $L^2 \wedge^r (G/K) \equiv L^2(G, K, \tau_r)$.

Theorem 3.3. *Let m be the dimension of G/K. Let $L^2(G, K, \tau_r)_d$ denote the discrete part of $L^2(G, K, \tau_r)$. Then*

$$L^2(G, K, \tau_r)_d = \begin{cases} \{0\} & \text{if } r \neq \dfrac{m}{2}, \\ \displaystyle\bigoplus_{j=1}^{l} \pi_{w_j \cdot \delta_G} & \text{if } r = \dfrac{m}{2}, \end{cases}$$

where the $\pi_{w_j \cdot \delta_G}$ are exactly the discrete series representations of G having trivial infinitesimal character, and each of them occurs with multiplicity 1. Moreover, the square integrable *harmonic forms on G/K consists exactly of the discrete series contribution to $L^2(G, K, \tau_r)$: each $\pi_{w_j \cdot \delta_G}$ is realized in $L^2(G, K, \tau_{m/2})$ on the null space for the Casimir operator Ω acting on $L^2(G, K, \tau_{\mu_j})$, where τ_{μ_j} is the multiplicity free irreducible subrepresentation of $\tau_{m/2}$ with highest weight $\mu_j = w_j \cdot 2\delta_G - 2\delta_K$ and is the minimal K-type of $\pi_{w_j \cdot \delta_G}$.*

In this result, it is understood that $-\Delta$ is realized by the action of the Casimir operator on $C^\infty(G, K, \tau_r)$: see *Kuga's formula* in [4, Ch. II, Theorem 2.5]. It follows then also that the Laplacian Δ has no (discrete) eigenvalue on $L^2 \wedge^r (G/K)$, except 0 (with infinite multiplicity) when $r = m/2$. The entire spectrum of Δ in our setting will be given in Section 4.

Let us now apply the theorem to our case $G/K = H^n(\mathbb{C})$. As before, let $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ be the Cartan subalgebra constituted with diagonal matrices. With notations above,

$$R_G := R(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) = \{\varepsilon_i - \varepsilon_j,\ 1 \le i \ne j \le n+1\},$$

while R_K and R_K^+ were, respectively, defined in (3.5) and (3.7). The Weyl group W_G (resp. W_K) is the group of permutations of $n+1$ (resp. of n) elements. Therefore, there are $l = n+1$ positive subsystems in R_G that are compatible with R_K^+. Choose

$$R_G^+ = \{\varepsilon_i - \varepsilon_j,\ 1 \le i < j \le n+1\}.$$

Then all the compatible positive systems in R_G are obtained as follows (see [2, Section 11]). Denote by s_β the reflection through the root β and put

$$w_j = \prod_{k=j+1}^{n} s_{\varepsilon_k - \varepsilon_{n+1}} (0 \le j \le n-1), \qquad w_n = \mathrm{id}.$$

Then $W_K \backslash W_G = \{W_K \cdot w_j,\ 0 \le j \le n\}$ and the $n+1$ positive systems in R_G that are compatible with R_K^+ are exactly the $w_j \cdot R_G^+$, with $0 \le j \le n$. The sums of positive roots are:

$$2\delta_G = \sum_{k=1}^{n+1} (n + 2 - 2k)\varepsilon_k, \tag{3.12}$$

$$2\delta_K = \sum_{k=1}^{n} (n + 1 - 2k)\varepsilon_k. \tag{3.13}$$

In view of Theorem 3.5, we must determine now the highest weights μ_j of the minimal K-types τ_{μ_j} of the discrete series $\pi_{w_j \cdot \delta_G}$. Again, by Lemma 11.16 in [2], we have

$$\mu_j = \sum_{k=1}^{j} \varepsilon_k - \sum_{k=j+1}^{n} \varepsilon_k + (n - 2j)\varepsilon_{n+1} \quad (0 \le j \le n).$$

In other words,

$$\tau_{\mu_j} = \tau'_{n-j,j} \quad (0 \le j \le n).$$

This identification completes the proof of the following result.

Theorem 3.4. *Let $0 \le r \le n$. The discrete part of the Plancherel formula for the space $L^2(G, K, \tau_r)$ of L^2 differential r-forms on $H^n(\mathbb{C})$ is given by*

$$L^2(G, K, \tau_r)_d = \begin{cases} \{0\} & \text{if } r \neq n, \\ \bigoplus_{p+q=n} L^2(G, K, \tau'_{p,q})_\Delta & \text{if } r = n, \end{cases}$$

where $L^2(G, K, \tau'_{p,q})_\Delta$ denotes the harmonic part of $L^2(G, K, \tau'_{p,q})$ and coincides with the Hilbert space of the discrete series representation $\pi_{w_q \cdot \delta_G}$.

Remarks.

1. *This theorem was first proved in* [4], *where all L^2 $(\mathfrak{g}, K)$- cohomology groups were calculated for real rank one semisimple Lie groups* (*Ch. VI, Theorem* 3.8), *and in particular for $G = SU(n, 1)$* (*Ch. VI, Theorem* 4.11). *In its weak form, i.e. as a (non)vanishing theorem for the L^2 cohomology on $H^n(\mathbb{C})$, the theorem can also be derived from results stated in more general settings by geometers: Donnelly and Fefferman* (*Theorem* 1.1 *in* [12]) *for the case of pseudoconvex domains in $\mathbb{C}^n$, and Gromov* ([15, *Theorem* 1.2.B]) *for the extension to the case of Kähler hyperbolic manifolds.*
2. *The theorem implies that L^2 harmonic forms are automatically primitive. Indeed, recall from Proposition* 2.1 *the Lefschetz decomposition*

$$\tau_n = \bigoplus_{p+q=n} \bigoplus_{k=0}^{\min(p,q)} \tau'_{p-k,q-k}.$$

Comparing with the theorem, we see that, for each fixed p (and $q = n - p$), the K-types $\tau'_{p-k,q-k}$ do not produce harmonic n-forms, except when $k = 0$. Roughly speaking, this is due to the fact that these K-types are 'too small' to be minimal K-types of discrete series with trivial infinitesimal character. Another (and simpler) explanation was given in [21], *Corollary* 2.5: *since L^* is bounded and commutes with Δ, L^* maps L^2 harmonic n-forms into L^2 harmonic $n - 2$ forms, i.e., into zero by the theorem.*
3. *The decomposition of $L^2(G, K, \tau_r)$ into irreducible components* (*Theorems* 3.2 *and* 3.4) *is realized by an appropriate Fourier transform. This will be developed in Section* 5.

It is well known that discrete series of G can be embedded into nonunitary principal series of G: this fact is a consequence of Casselman's Subrepresentation Theorem (see e.g. [23, Theorem 8.37]). As a complement of Theorem 3.4, we list explicitly the principal series representations in which our discrete series representations $\pi_{w_q \cdot \delta_G}$ ($0 \le q \le n$) appear as unitary components. Remind that two admissible representations are said infinitesimally equivalent if the subspaces of their respective K-finite vectors are isomorphic as $(\mathfrak{g}, K)$-modules. The proof of the following result derives from the tables in [2, Section 18].

Proposition 3.5. *Let $p + q = n$.*

(i) $\pi_{w_0 \cdot \delta_G}$ *is infinitesimally equivalent with a unitary submodule of* $\pi_{\sigma'_{n-1,0}, -i}$;

(ii) *if* $1 \leq q \leq n-1$, $\pi_{w_q \cdot \delta_G}$ *is infinitesimally equivalent with a unitary submodule of* $\pi_{\sigma'_{p-1,q}, -i}$ *and of* $\pi_{\sigma'_{p,q-1}, -i}$;
(iii) $\pi_{w_n \cdot \delta_G}$ *is infinitesimally equivalent with a unitary submodule of* $\pi_{\sigma'_{0,n-1}, -i}$.
In all cases, the discrete series submodules appear with multiplicity 1.

Remark. *An equivalent formulation of the proposition would be that the discrete series representations are infinitesimally equivalent with subquotients of certain nonunitary principal series (namely, the ones with the same parameter* σ *and with the opposite parameter* $+\mathrm{i}$ *instead of* $-\mathrm{i}$*). This follows from the existence of a natural duality between* $\pi_{\sigma,\lambda}$ *and* $\pi_{\sigma,\bar{\lambda}}$ *(see e.g.* [23, *Eq.* (8.114)]).

4. The spectrum of the Hodge–de Rham Laplacian

We keep the notations of the previous sections. Let B denote the Killing form on $\mathfrak{g} = \mathfrak{su}(n,1)$. It is standard that

$$B(X,Y) = 2(n+1)\,\mathrm{tr}(XY) \quad (X, Y \in \mathfrak{g}).$$

In order to recover the normalization of the inner product on $\mathfrak{p}$ we made in (2.2), we put

$$\langle\cdot,\cdot\rangle = \frac{1}{B(H_0,H_0)} B(\cdot,\cdot) = \frac{1}{4(n+1)} B(\cdot,\cdot), \tag{4.1}$$

where H_0 was defined in (3.1). Let Ω be the *Casimir operator* associated with the bilinear form on $\mathfrak{g}$ defined by (4.1), i.e.

$$\Omega = \sum_{i,j} a^{ij} X_i X_j, \tag{4.2}$$

where (X_i) is any basis of $\mathfrak{g}$ and (a^{ij}) is the inverse of the matrix with coefficients $a_{ij} = \mathrm{Re}\langle X_i, X_j\rangle$. [1] Recall the identification (Kuga's formula)

$$\Delta \equiv -\Omega$$

on the space $C^\infty(G,K,\tau_r)$ of smooth differential r-forms on $H^n(\mathbb{C})$. Thanks to the Plancherel theorem for $L^2(G,K,\tau_r)$ (i.e., Theorems 3.2 and 3.4), the L^2 spectrum of the Laplacian is obtained by studying the action of the Casimir operator on the various components occuring in the Plancherel formula. As was remarked in Theorem 3.4, discrete series representations with trivial infinitesimal character produce harmonic forms. On the other hand, it is well-known that Ω is a central operator in the enveloping algebra of $\mathfrak{g}$ and that $\pi_{\sigma,\lambda}(\Omega)$ is a scalar operator acting by a constant. More precisely, it follows from [23], Proposition 8.22 and Lemma 12.28, that

$$\pi_{\sigma,\lambda}(\Omega) = (\langle \Lambda_\sigma + \mathrm{i}\lambda, \Lambda_\sigma + \mathrm{i}\lambda\rangle - \langle\delta_G,\delta_G\rangle)\mathrm{Id}, \tag{4.3}$$

[1] The same notation was introduced in Section 2 for the fundamental form on $H^n(\mathbb{C})$, but this object will not be used anymore in the sequel.

where Λ_σ is the infinitesimal character of σ. Let μ_σ be the highest weight of σ and $2\delta_M = \sum_{k=2}^{n}(n+2-2k)\varepsilon_k$ denote the sum of the roots in R_M^+. Then $\Lambda_\sigma = \mu_\sigma + \delta_M$, and an easy calculation gives

$$\pi_{\sigma,\lambda}(\Omega) = -(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle - \langle\mu_\sigma, \mu_\sigma + 2\delta_M\rangle)\mathrm{Id}.$$

Lemma 4.1. *Let* $0 \le p+q \le n-1$. *Then the identity*

$$\pi_{\sigma'_{p,q},\lambda}(\Omega) = -[\lambda^2 + (n-p-q)^2]\mathrm{Id}$$

holds on the space $C^\infty(G, P, \sigma'_{p,q} \otimes \mathrm{e}^{\mathrm{i}\lambda} \otimes \mathbf{1})$.

Proof. It is an easy consequence of (3.10). □

Remark. *The same result was observed in* [6, *Lemma* 4.12].

We state now the main result of this section.

Corollary 4.2. *For* $0 \le p+q \le 2n$, *denote by* $\mathrm{spec}\Delta_{p,q}$ *the* L^2 *spectrum of the Hodge–de Rham Laplacian on* (p,q)*-forms on* $H^n(\mathbb{C})$. *Then*

$$\mathrm{spec}\Delta_{p,q} = \begin{cases} [(n-p-q)^2, +\infty[& \text{if } p+q \neq n, \\ \{0\} \cup [1, +\infty[& \text{if } p+q = n. \end{cases}$$

Proof. Consider first the restriction $\Delta'_{p,q}$ of $\Delta_{p,q}$ to primitive forms. The eigenvalues of $\Delta'_{p,q}$ are given by Lemma 4.1 for $\sigma \in \widehat{M}(\tau'_{p,q})$. When $p+q=n$, as was noticed in Section 3.2, we must add the eigenvalue 0 coming from the action of discrete series. Using Theorems 3.2 and 3.4, and observing the contributions of the various $\sigma \in \widehat{M}(\tau'_{p,q})$ in each case, we see that the eigenvalues of $\Delta'_{p,q}$ are

$$\begin{cases} \{\lambda^2 + (n-p-q)^2,\ \lambda \in \mathbb{R}\} & \text{if } 0 \le p+q \le n-1, \\ \{0\} \cup \{\lambda^2+1,\ \lambda \in \mathbb{R}\} & \text{if } p+q = n. \end{cases} \tag{4.4}$$

Since Δ preserves the Lefschetz decomposition for $\tau_{p,q}$ (Proposition 2.1), i.e.,

$$\Delta_{p,q} = \sum_{k=0}^{\min(p,q)} L^k \Delta'_{p-k,q-k},$$

and since the main contribution to the spectrum is given by $\Delta'_{p,q}$, i.e.,

$$\mathrm{spec}\,\Delta'_{p,q} \le \mathrm{spec}\,\Delta'_{p-1,q-1} \le \dots,$$

it follows that (4.4) gives the exact L^2 spectrum of $\Delta_{p,q}$ for $0 \le p+q \le n$. Then we use the Hodge duality $\mathrm{spec}\,\Delta_{p,q} = \mathrm{spec}\,\Delta_{n-q,n-p}$. □

Remark. *Similar considerations lead easily to the determination of the spectrum of the Bochner Laplacian, by using an explicit Weitzenböck formula. We omit details.*

When considering functions (i.e. 0-forms) on symmetric spaces G/K, there is a classical way to construct eigenfunctions of the Laplacian (and even of the full algebra $\mathbb{D}(G/K)$ of all invariant differential operators), which consists in taking *Poisson transforms* of hyperfunctions on the boundary K/M of G/K. This statement is known as *Helgason's conjecture* and was first proved in [16] (rank 1 case) and [22] (general case). In the case of homogeneous vector bundles $G\times_K V_\tau$, the extension of Helgason's conjecture was examined in [13,14] (for the differential forms bundle over real and complex hyperbolic spaces), [38] (for vector bundles over rank one symmetric spaces G/K), [30,40] (for vector bundles over general symmetric spaces G/K). See also [32] for the construction of Poisson transforms in the particular case of the bundle of differential forms over real hyperbolic spaces, and for other references in this particular setting.

In this paper, our purpose, much more modest and easy, is only to exhibit sections of a certain vector bundle over the boundary $G/P=K/M=S(\mathbb{C}^n)$ of $G/K=H^n(\mathbb{C})$ whose Poisson transforms are eigenforms for the Hodge–de Rham Laplacian.

As before, we can and shall restrict to primitive forms. In the sequel, fix $\sigma=\sigma'_{p,q}$ with $0\le p+q\le n-1$. For $\tau\in\widehat{K}$ such that $\sigma\in\widehat{M}(\tau)$ (generically, τ is one of $\tau'_{p,q},\tau'_{p+1,q},\tau'_{p,q+1},\tau'_{p+1,q+1}$), denote by P_σ^τ the generator of the one-dimensional space $\mathrm{Hom}_K(\mathcal{H}_{\sigma,\lambda},V_\tau)\simeq\mathrm{Hom}_K(L^2(K,M,\sigma),V_\tau)$ (for any $\lambda\in\mathbb{C}$) defined by:

$$P_\sigma^\tau(f):=\sqrt{\frac{\dim\tau}{\dim\sigma}}\int_K \mathrm{d}k\,\tau(k)f(k),\quad \forall f\in C^\infty(G,P,\sigma\otimes\mathrm{e}^{\mathrm{i}\lambda}\otimes\mathbf{1}),\tag{4.5}$$

where $\mathrm{d}k$ is the Haar measure on K normalized by $\int_K \mathrm{d}k=1$.[2] In other words, P_σ^τ is a K-equivariant orthogonal projection of $\mathcal{H}_{\sigma,\lambda}$ onto V_τ. For $\lambda\in\mathbb{C}$ and $f\in C^\infty(G,P,\sigma\otimes\mathrm{e}^{\mathrm{i}\lambda}\otimes\mathbf{1})$, set:

$$\phi_{\sigma,\lambda}^\tau(x)(f):=P_\sigma^\tau\circ\pi_{\sigma,\lambda}(x^{-1})f\quad(\forall x\in G).$$

Then it is easily checked that the function $x\mapsto\phi_{\sigma,\lambda}^\tau(x)(f)$ is an element of $C^\infty(G,K,\tau)$, i.e., is a primitive differential form on G/K. Moreover, since the map

$$\begin{aligned}C^\infty(G,P,\sigma\otimes\mathrm{e}^{\mathrm{i}\lambda}\otimes\mathbf{1})&\longrightarrow C^\infty(G,K,\tau)\\ f&\longmapsto\phi_{\sigma,\lambda}^\tau(\cdot)(f)\end{aligned}$$

is continous, linear and G-equivariant, we call it the *Poisson transform* on $C^\infty(G,P,\sigma\otimes\mathrm{e}^{\mathrm{i}\lambda}\otimes\mathbf{1})$.

Proposition 4.3. *Fix $\sigma=\sigma'_{p,q}$ with $0\le p+q\le n-1$ and let $\tau\in\widehat{K}$ be such that $\sigma\in\widehat{M}(\tau)$. Since τ is a K-type associated with a primitive differential forms bundle on $H^n(\mathbb{C})$, denote by Δ_τ the restriction of the Hodge–de Rham Laplacian to $C^\infty(G,K,\tau)$. Then, for any $\lambda\in\mathbb{C}$, $f\in C^\infty(G,P,\sigma\otimes\mathrm{e}^{\mathrm{i}\lambda}\otimes\mathbf{1})$ and $x\in G$,*

$$\Delta_\tau\phi_{\sigma,\lambda}^\tau(x)(f)=[\lambda^2+(n-p-q)^2]\phi_{\sigma,\lambda}^\tau(x)(f).$$

[2] The integers $\dim\tau$ and $\dim\sigma$ will be computed in Lemma 6.2.

Proof. We apply Kuga's formula and Lemma 4.1:

$$\begin{aligned} \Delta_\tau \phi^\tau_{\sigma,\lambda}(x)(f) &= -\phi^\tau_{\sigma,\lambda}(x : \Omega)(f) = -P^\tau_\sigma \circ \pi_{\sigma,\lambda}(\Omega) \circ \pi_{\sigma,\lambda}(x^{-1}) f \\ &= [\lambda^2 + (n-p-q)^2]\, \phi^\tau_{\sigma,\lambda}(x)(f). \qquad \square \end{aligned}$$

Remark. *Julg and Kasparov* ([21, *Section* 2]) *have noticed that elements of* $C^\infty(G, P, \sigma \otimes e^{i\lambda} \otimes \mathbf{1})$ *can be identified with Rumin differential forms on the contact manifold* K/M (*see* [34] *or* [35] *for a general definition of the Rumin complex*). [3] *In the same article, Section* 4, *the authors construct also very explicitly a Poisson transform* (*that they call 'Szegö map'*) *which sends isomorphically Rumin* $(n-1)$*-forms on* K/M *to* L^2 *harmonic* n*-forms on* $H^n(\mathbb{C})$.

5. Basic Fourier analysis

5.1. Spherical functions and spherical Fourier transform of radial functions

In [32], Section 3 [4] (see also [9,10]), we showed that the spherical harmonic analysis on homogeneous vector bundles $G \times_K V_\tau$ over noncompact Riemannian symmetric spaces G/K, when τ is irreducible and unitary, can be carried out similarly to the classical 'scalar case' (i.e., when $\tau = \mathbf{1}$, see e.g. [17, Ch. IV]), provided that the triple (G, K, τ) verify a certain condition. More precisely, denote by $\Gamma(G, K, \tau, \tau)$ the set of τ*-radial functions on* G, i.e., of functions $F : G \to \operatorname{End} V_\tau$ verifying the double K-equivariance condition:

$$F(k_1 x k_2) = \tau(k_2)^{-1} F(x) \tau(k_1)^{-1} \qquad (\forall x \in G,\ \forall k_1, k_2 \in K).$$

We say that (G, K, τ) is a *Gelfand triple* if, endowed with the convolution product

$$(F * H)(x) = \int_G \mathrm{d}y\, F(y^{-1}x) H(y) = \int_G \mathrm{d}y\, F(y) H(xy^{-1}),$$

the algebra $C_c(G, K, \tau, \tau)$ of continuous τ-radial functions on G with compact support is *commutative*. Once this condition is assumed for the triple (G, K, τ), one can develop a very satisfying theory for τ*-spherical functions on* G, i.e., for spherical functions attached to the vector bundle $G \times_K V_\tau$ over G/K (actually, this theory is well-adapted also in a more general setting). Moreover, the identification of Gelfand triples associated with vector bundles $G \times_K V_\tau$ is highly facilitated by a criterion which is essentially due to Deitmar (see [11]; other references and historical comments are given in [31, Section 5.1]). In particular, (G, K, τ) is a Gelfand triple if and only if $\tau_{|M}$ is multiplicity free or if and only

[3] Recall that, in the case of real hyperbolic spaces, the corresponding elements of $C^\infty(G, P, \sigma \otimes e^{i\lambda} \otimes \mathbf{1})$ are identified to (standard) differential forms on K/M (see e.g. [14, Section 8] or [32, Section 2]).

[4] In the sequel, we shall frequently quote results from references [32] or [33]. At this time, the proofs of these results can only be found in [31].

if the algebra $\mathbb{D}(G, K, \tau)$ of left-invariant differential operators acting on $C^\infty(G, K, \tau)$ is commutative.

Assume (G, K, τ) is a Gelfand triple, and let $\Phi \in C^\infty(G, K, \tau, \tau)$ with $\Phi(e) = \mathrm{Id}$. Then Φ is *a τ-spherical function on G* if Φ is an eigenfunction for the algebra $\mathbb{D}(G, K, \tau)$, in the sense that there exists a character χ_Φ of $\mathbb{D}(G, K, \tau)$ such that

$$D\Phi(\cdot)\xi = \chi_\Phi(D)\Phi(\cdot)\xi$$

for any $D \in \mathbb{D}(G, K, \tau)$ and for one nonzero $\xi \in V_\tau$ (hence for all $\xi \in V_\tau$). The set of τ-spherical functions on G will be denoted by $\Sigma(G, K, \tau, \tau)$.

Actually, τ-spherical functions on G have three other equivalent characterizations: as characters of the convolution algebra $C_c(G, K, \tau, \tau)$, as solutions of functional equations and as eigenfunctions with respect to convolution with $C_c(G, K, \tau, \tau)$ (see [32, Section 3]).

Assume from now on that $G = SU(n, 1)$ and $K = S(U(n) \times U(1))$, and fix $\tau = \tau'_{p,q}$ with $0 \le p + q \le n$. By Proposition 3.1, (G, K, τ) is then a Gelfand triple (in fact, this would be the case for any $\tau \in \widehat{K}$ when $G = SU(n, 1)$, see [26] or [11]) and we can apply the τ-spherical function theory to our setting. Let us introduce first some more notation.

Using the Iwasawa decomposition $G = KAN$, we denote by H the Iwasawa projector on A such that $H(ka_t n) = t$ and by $\underline{k}$ the projector on K. Then a principal series representation $\pi_{\sigma,\lambda}$ acts on $\mathcal{H}_{\sigma,\lambda|K} \simeq L^2(K, M, \sigma)$ by

$$\pi_{\sigma,\lambda}(x)f(k) = \mathrm{e}^{-(\mathrm{i}\lambda+\rho)H(x^{-1}k)} f(\underline{k}(x^{-1}k)) \quad (\forall x \in G,\ \forall k \in K).$$

For $\sigma \in \widehat{M}(\tau)$, define P_σ^τ as in (4.5). Put $J_\sigma^\tau = (P_\sigma^\tau)^*$, i.e., J_σ^τ is the generator of the one-dimensional space $\mathrm{Hom}_K(V_\tau, \mathcal{H}_{\sigma,\lambda}) \simeq \mathrm{Hom}_K(V_\tau, L^2(K, M, \sigma))$ defined by

$$J_\sigma^\tau\xi = \sqrt{\frac{\dim\tau}{\dim\sigma}}\, P_\sigma \circ \{\tau(\cdot)^{-1}\xi\},$$

where P_σ denotes the orthogonal projection of V_τ onto its σ-isotypical component $V_\tau(\sigma) \simeq V_\sigma$. For $\sigma \in \widehat{M}(\tau)$ and $\lambda \in \mathbb{C}$, it is clear that the map

$$x \mapsto \Phi_{\sigma,\lambda}^\tau(x) = P_\sigma^\tau \circ \pi_{\sigma,\lambda}(x^{-1}) \circ J_\sigma^\tau \tag{5.1}$$

defines a τ-radial function on G.

Theorem 5.1. *For $\tau = \tau'_{p,q}$, $\sigma \in \widehat{M}(\tau)$ and $\lambda \in \mathbb{C}$, set $\Phi_{\sigma,\lambda}^\tau$ as in* (5.1). *Define Δ_τ as in Proposition* 4.4. *Then:*

(i) $\Phi_{\sigma,\lambda}^\tau \in \Sigma(G, K, \tau, \tau)$. *Moreover, for any $\xi \in V_\tau$,*

$$\Delta_\tau\Phi_{\sigma,\lambda}^\tau(\cdot)\xi = [\lambda^2 + (n - k_\sigma)^2]\,\Phi_{\sigma,\lambda}^\tau(\cdot)\xi, \tag{5.2}$$

with

$$k_\sigma = \begin{cases} p+q & \text{if } \sigma = \sigma'_{p,q}, \\ p+q-1 & \text{if } \sigma = \sigma'_{p-1,q} \text{ or } \sigma = \sigma'_{p,q-1}, \\ p+q-2 & \text{if } \sigma = \sigma'_{p-1,q-1}. \end{cases} \tag{5.3}$$

(ii) $\Phi^{\tau}_{\sigma,\lambda}$ *admits the following representation as Eisenstein integral:*

$$\Phi^{\tau}_{\sigma,\lambda}(x) = \frac{\dim\tau}{\dim\sigma}\int_K \mathrm{d}k\mathrm{e}^{-(\mathrm{i}\lambda+\rho)H(xk)}\tau(k)\circ P_\sigma\circ\tau(\underline{k}(xk)^{-1}). \tag{5.4}$$

In particular, $\Phi^{\tau}_{\sigma,\lambda}$ *is holomorphic in the variable* λ.

(iii) $\Sigma(G,K,\tau,\tau) = \{\Phi^{\tau}_{\sigma,\lambda}\ :\ \sigma\in\widehat{M}(\tau),\ \lambda\in\mathbb{C}/\{\pm1\}\}$, *i.e.,* all τ*-spherical functions on* G *are associated with principal series* $\pi_{\sigma,\lambda}$ *with* $\sigma\in\widehat{M}(\tau)$ *and* $\lambda\in\mathbb{C}/\{\pm1\}$.

Proof. The fact that $\Phi^{\tau}_{\sigma,\lambda}$ is τ-spherical follows from [32, Proposition 7]. The eigenvalues description for the Laplacian follows from Proposition 4.3, once we remark the identity $\Phi^{\tau}_{\sigma,\lambda}(\cdot)\xi = \phi^{\tau}_{\sigma,\lambda}(\cdot)(J^{\tau}_{\sigma}\xi)$. Assertions (ii) and (iii) are, respectively, Propositions B.14 in [31] and Proposition 9 in [32]. □

Remarks.

1. *The unitarity of the principal series representations implies the identity* $\Phi^{\tau}_{\sigma,\lambda}(x)^* = \Phi^{\tau}_{\sigma,\bar{\lambda}}(x^{-1})$ *for any* $x\in G$ *and* $\lambda\in\mathbb{C}$.
2. *The Weyl group action on the same representations yields the relation* $\Phi^{\tau}_{\sigma,\lambda} = \Phi^{\tau}_{\sigma,-\lambda}$ *for all* $\lambda\in\mathbb{C}$ *(see Section* 3.1).

It was remarked in [33], Section 4 (see also [9]), that the spherical Fourier transform of τ-radial functions on G is naturally defined as the *Gelfand transform* of these functions (actually, the set $\Sigma(G,K,\tau,\tau)$ of τ-spherical functions can be defined as the Gelfand spectrum of $C_c(G,K,\tau,\tau)$). In our setting, according to item (iii) in the previous result, for $F\in C_c(G,K,\tau,\tau)$, we define a spherical transform $\mathcal{H}^{\tau}_{\sigma,\lambda}(F)$ of F associated with each principal series representation $\pi_{\sigma,\lambda}$ such that $\sigma\in\widehat{M}(\tau)$ by

$$\mathcal{H}^{\tau}_{\sigma,\lambda}(F) := \frac{1}{\dim\tau}\int_G \mathrm{d}x\,\mathrm{tr}\{F(x)\Phi^{\tau}_{\sigma,\lambda}(x^{-1})\}\in\mathbb{C}. \tag{5.5}$$

Here, and thereafter in Section 5, *the measures on G and K are normalized as in Section* 6. *The spherical Fourier transform of F* is, by definition, the collection of (even) functions on $\mathbb{C}$:

$$\lambda\mapsto\{\mathcal{H}^{\tau}_{\sigma,\lambda}(F)\}_{\sigma\in\widehat{M}(\tau)}.$$

We can state now the main result of this section, namely, the inversion formula for the spherical Fourier transform on $C^{\infty}_c(G,K,\tau,\tau)$.

Theorem 5.2. *Let* $\tau = \tau'_{p,q}$ *for* $0\le p+q\le n$. *The spherical Fourier transform on* $C^{\infty}_c(G,K,\tau,\tau)$ *is inverted by the following formulas:*

(i) *if* $0\le p+q\le n-1$, *for any* $x\in G$,

$$F(x) = \sum_{\sigma\in\widehat{M}(\tau)} c_\sigma\int_0^{+\infty}\mathrm{d}\nu_\sigma(\lambda)\,\mathcal{H}^{\tau}_{\sigma,\lambda}(F)\,\Phi^{\tau}_{\sigma,\lambda}(x), \tag{5.6}$$

where $c_\sigma > 0$ is a normalizing constant depending only on σ.

(ii) *if $p+q=n$, the following discrete term must be added to the right-hand side of* (5.6):

$$c'_\sigma\, \mathcal{H}^\tau_{\sigma,\pm i}(F)\Phi^\tau_{\sigma,\pm i}(x), \tag{5.7}$$

where σ is a selected $\widehat{M}(\tau)$-parameter such that the discrete series $\pi_{w_q\cdot\delta_G}$ is infinitesimally equivalent with a subrepresentation of the nonunitary principal series $\pi_{\sigma,-i}$ (see Proposition 3.5*), and $c'_\sigma > 0$ is a constant depending only on σ.*

Remark. *Although this theorem is a particular case of Eq.* (46) *in* [9], *it deserves some comment, notably in case (ii). The τ-spherical function $\Phi^\tau_{\sigma,\pm i}$ occuring in* (5.7) *is harmonic by* (5.2)*, and the whole discrete term* (5.7) *represents the harmonic component of F, exactly as in Theorem* 3.4*. Actually, if one associates to the discrete series $\pi_q := \pi_{w_q\cdot\delta_G}$ a τ-spherical function $\Phi^\tau_{\pi_q}$ as in* (5.1)*, one should observe that (generically for q)*

$$\Phi^\tau_{\pi_q} = \Phi^\tau_{\sigma'_{p-1,q},\pm i} = \Phi^\tau_{\sigma'_{p,q-1},\pm i}$$

(and that these functions are L^2), so that the discrete term (5.7) *could be rewritten as*

$$c'_{\pi_q}\, \mathcal{H}^\tau_{\pi_q}(F)\, \Phi^\tau_{\pi_q}(x),$$

which makes the comparison with Theorem 3.4 *more clear. On the other hand, the Plancherel measure $\mathrm{d}\nu_\sigma(\lambda)$ (with σ as in the statement (ii) of the theorem) has its first pole on the positive imaginary axis at $\lambda_0 = i$ (all poles of $\mathrm{d}\nu_\sigma$ are listed in* (6.1)*), and the discrete term should arise then from the residue calculus of this measure at λ_0 after the integration contour is shifted 'up'. It is understood that this discussion is not a demonstration. However, it is strongly suggested by the comparison with a similar phenomenon that occurs in the case of the bundle of differential forms over real hyperbolic spaces (see Theorem* 6.15 *and Proposition* 6.24 *in* [31] *or* [33, *Section* 4]).

Thanks to the Lefschetz decomposition for τ_r (Proposition 2.1), it is clear that the theorem above extends immediately to radial functions $F \in C^\infty_c(G, K, \tau_r, \tau_r)$. On the other hand, the *Plancherel formula* for $L^2(G, K, \tau_r, \tau_r)$ follows from the inversion formula by standard arguments (see e.g. [31, Section 6]), and we omit the details. Finally, let us mention that a Paley–Wiener Theorem for the spherical transform can be borrowed, in theory, from Campoli's results ([7, Section 3.2.1]). However, the result is not very readable as long as we do not know very explicitly the τ-spherical functions.

5.2. *Fourier transform of differential forms*

In this section, we introduce the Fourier transform of (primitive) differential forms on $H^n(\mathbb{C})$ and we deduce from Theorem 5.2 the inversion and Plancherel formulas for this transform.

As usual, fix $\tau = \tau'_{p,q}$ with $0 \le p+q \le n$. Remind that the Haar measure on G and K are normalized as in Section 6. With the same notation as before, if $f \in \Gamma(G, K, \tau)$ and

π is a (principal series or discrete series) representation occuring in the decomposition of $L^2(G, K, \tau'_{p,q})$ (see Section 3), set

$$\mathcal{H}^\tau_\pi(f) := \frac{1}{\dim\tau} \int_G \mathrm{d}x\, \pi(x) \circ J^\tau_\pi\, f(x) \in \mathcal{H}_\pi \tag{5.8}$$

(whenever this integral converges). If $\pi = \pi_{\sigma,\lambda}$ is a principal series representation, we shall write $\mathcal{H}^\tau_{\sigma,\lambda}(f)$ instead of $\mathcal{H}^\tau_{\pi_{\sigma,\lambda}}(f)$. Note that, in this case, $\mathcal{H}^\tau_{\sigma,\lambda}(f)$ is an element of $C^\infty(K, M, \sigma)$ if, for instance, f is taken in $C^\infty_c(G, K, \tau)$, which we shall suppose from now on for convenience. The *Fourier transform of* $f \in C^\infty_c(G, K, \tau)$ is, by definition, the collection of $C^\infty(K, M, \sigma)$-valued functions

$$\lambda \mapsto \{\mathcal{H}^\tau_{\sigma,\lambda}(f)\}_{\sigma\in\widehat{M}(\tau)}.$$

Remarks.

1. *Definition* (5.8) *is motivated by an argument given in* [31, *Section* 6.3]. *Camporesi* [8] *uses a slightly different one. The comparison between our respective definitions is facilitated by the following expansion:*

$$\mathcal{H}^\tau_{\sigma,\lambda}(f)(k) = \frac{1}{\sqrt{\dim\tau\cdot\dim\sigma}} \int_G \mathrm{d}x\, \mathrm{e}^{-(\mathrm{i}\lambda+\rho)H(x^{-1}k)} P_\sigma\, \tau(\underline{k}(x^{-1}k)^{-1}) f(x).$$

2. *We have introduced in Section* 5.1 *the same notation for the spherical transform, but a* τ*-spherical function is always denoted with a capital letter, while a function of type* τ *is always denoted with a lower case one, which excludes any ambiguity.*

The inversion formula and the Plancherel Theorem for the Fourier transform can then be deduced from Theorem 5.2 (the inversion formula can be also taken as a particular case of [8], Theorem 1.1). Although the proofs are rather technical (especially in the case $p+q = n$), we shall omit them, since they are exactly the same as the ones given in [31] for differential forms on real hyperbolic spaces. Note that these results are exactly the analytic versions of the abstract Plancherel theorem established in Section 3.

Theorem 5.3. *Fix* $\tau = \tau'_{p,q}$ *with* $0 \le p+q \le n$. *The Fourier transform on* $C^\infty_c(G, K, \tau)$ *is inverted by the following formulas:*

(i) *if* $0 \le p+q \le n-1$, *for any* $x \in G$,

$$f(x) = \dim\tau \sum_{\sigma\in\widehat{M}(\tau)} c_\sigma \int_0^{+\infty} \mathrm{d}\nu_\sigma(\lambda)\, P^\tau_\sigma \circ \pi_{\sigma,\lambda}(x^{-1})\, \mathcal{H}^\tau_{\sigma,\lambda}(f). \tag{5.9}$$

(ii) *if* $p+q = n$, *the following discrete term must be added to the right-hand side of* (5.9)*:*

$$(\dim\tau)c'_\sigma\, P^\tau_\sigma \circ \pi_{\sigma,-i}(x^{-1})\mathcal{H}^\tau_{\sigma,-i}(f), \tag{5.10}$$

where σ is a selected $\widehat{M}(\tau)$-parameter such that the discrete series $\pi_{w_q \cdot \delta_G}$ is infinitesimally equivalent with a subrepresentation of the nonunitary principal series $\pi_{\sigma,-i}$ (see Proposition 3.5).

In both cases, the constants are the same as in Theorem 5.4.

Theorem 5.4. *Fix $\tau = \tau'_{p,q}$ with $0 \leq p+q \leq n$. Set, for short, $\pi_q := \pi_{w_q \cdot \delta_G}$ (see Theorem* 3.4) *and keep notations of Theorem* 5.3. *Then:*

(i) *if $0 \leq p+q \leq n-1$, for any $f \in C_c^\infty(G, K, \tau)$,*

$$\|f\|^2_{L^2(G,K,\tau)} = (\dim \tau)^2 \sum_{\sigma \in \widehat{M}(\tau)} c_\sigma \int_0^{+\infty} \mathrm{d}\nu_\sigma(\lambda) \|\mathcal{H}^\tau_{\sigma,\lambda}(f)\|^2_{L^2(K,M,\sigma)}, \tag{5.11}$$

and the Fourier transform extends to a bijective isometry from $L^2(G, K, \tau)$ onto

$$\bigoplus_{\sigma \in \widehat{M}(\tau)} L^2(\mathbb{R}_+, (\dim \tau)^2 c_\sigma \, \mathrm{d}\nu_\sigma; L^2(K, M, \sigma)).$$

(ii) *if $p+q = n$, the discrete term*

$$(\dim \tau)^2 c'_\sigma \|\mathcal{H}^\tau_{\pi_q}(f)\|^2_{\mathcal{H}_{\pi_q}}$$

must be added to the right-hand side of (5.11), *and the Fourier transform extends to a bijective isometry from $L^2(G, K, \tau)$ onto*

$$\left[\bigoplus_{\sigma \in \widehat{M}(\tau)} L^2(\mathbb{R}_+, (\dim \tau)^2 c_\sigma \, \mathrm{d}\nu_\sigma; L^2(K, M, \sigma))\right] \oplus (\dim \tau)^2 c'_\sigma \mathcal{H}_{\pi_q}.$$

Remarks.

1. *Obviously, Proposition* 2.1 *induces similar results for all (not necessarily primitive) differential r-forms on $H^n(\mathbb{C})$.*
2. *The dimension of $\tau = \tau'_{p,q}$ will be given in Lemma* 6.2.

6. The Plancherel measure

In this section, we give an explicit expression for the continuous Plancherel measure $\mathrm{d}\nu_{\sigma'_{p,q}}(\lambda)$ on $\mathbb{R}$ that appeared in Theorem 3.2 and in results of Section 5. This expression will also be useful in Section 7.

In [28], Theorem 3.1(iii), were given the Plancherel measures $\mathrm{d}\nu_\sigma(\lambda)$ associated with all principal series representations $\pi_{\sigma,\lambda}$ of the group $G = SU(n, 1)$. Thus, our purpose is to specialize Miatello's result to the principal series representations that contribute to the continuous part of the Plancherel formula. Obviously, it suffices to consider the M-types $\sigma = \sigma'_{p,q}$.

Let us fix some more notation and precise normalizations of Haar measures. Since $d\nu_\sigma(\lambda)$ is known to be absolutely continuous with respect to the Lebesgue measure $d\lambda$ on $\mathbb{R}$, we shall write

$$d\nu_\sigma(\lambda) = \nu_\sigma(\lambda) \cdot d\lambda.$$

Then ν_σ can be viewed as a meromorphic function on $\mathbb{C}$, with poles on the imaginary axis. Moreover, when restricted to $\mathbb{R}$, ν_σ is a non-negative even function.

Let θ be the Cartan involution on $\mathfrak{g}$ associated with $\mathfrak{k}$. Then

$$\langle X, Y\rangle = -\frac{1}{4(n+1)} B(X, \theta Y)$$

defines an inner product on $\mathfrak{g}$ (the constant has the same meaning as in (4.1)). This inner product induces Riemannian measures $d\tilde{x}$ and $d\tilde{k}$ on G and K, respectively. If $\operatorname{vol} K$ is the volume of K with respect to $d\tilde{k}$, put $dx = (\operatorname{vol} K)^{-1}\, d\tilde{x}$ and $dk = (\operatorname{vol} K)^{-1}\, d\tilde{k}$.

In the sequel

$$\binom{a}{b} = \frac{a!}{b!(a-b)!}$$

denotes the usual binomial coefficient.

Theorem 6.1. *Let* $0 \le p+q \le n-1$. *Set* $m = \min(|-n-p+q|, |n-p+q|)$ *and* $M = \max(|-n-p+q|, |n-p+q|)$. *Then the Plancherel density associated with principal series representations* $\pi_{\sigma'_{p,q},\lambda}$ *is given by the following formulas:*

(i) *if* $n-p+q$ *is even, for any* $\lambda \in \mathbb{C}$,

$$\nu_{\sigma'_{p,q}}(\lambda) = \frac{2^{2-2n}}{\pi^{n+1} n!}(n-p-q)\binom{n}{p}\binom{n}{q} \times \left(\frac{\pi\lambda}{2}\coth\frac{\pi\lambda}{2}\right)\lambda^2 \prod_{\substack{k=2 \\ k \text{ even} \\ k \neq n-p-q}}^{m} (\lambda^2+k^2)^2 \prod_{\substack{k=m+2 \\ k \text{ even}}}^{M} (\lambda^2+k^2),$$

(ii) *if* $n-p+q$ *is odd, for any* $\lambda \in \mathbb{C}$,

$$\nu_{\sigma'_{p,q}}(\lambda) = \frac{2^{-2n}}{\pi^{n-1} n!}(n-p-q)\binom{n}{p}\binom{n}{q} \times \frac{\operatorname{th}\pi(\lambda/2)}{\pi\lambda/2}\lambda^2 \prod_{\substack{k=1 \\ k \text{ odd} \\ k \neq n-p-q}}^{m} (\lambda^2+k^2)^2 \prod_{\substack{k=m+2 \\ k \text{ odd}}}^{M} (\lambda^2+k^2).$$

In both cases, $\nu_{\sigma'_{p,q}}$ *has (simple) poles in the set*

$$S_{p,q} = \{\pm i(n-p-q)\} \cup \{\pm i(M+2+2l),\ l \in \mathbb{N}\}. \tag{6.1}$$

Proof. We recall first Theorem 3.1(iii) of Miatello [28], which gives the Plancherel density $\nu_\sigma(\lambda)$ associated with any $\sigma \in \widehat{M}$ when $G = SU(n, 1)$. Let σ have highest weight written in the form

$$\mu'_\sigma = \sum_{k=2}^{n-1} s_k\, \varepsilon_k + \frac{s}{2} \sum_{k=2}^{n} \varepsilon_k$$

(the ε_k's are defined as in (3.6)), with $s, s_2, \ldots, s_{n-1} \in \mathbb{Z}$ and $s_2 \geq \cdots \geq s_{n-1} \geq 0$. Set $s_n = 0$. Then

$$\nu_\sigma(\lambda) = \frac{1}{\pi^n\, \Gamma(n)} (\dim \sigma)\, \frac{\lambda}{2} \prod_{k=1}^{n-1} \left[\left(\frac{\lambda}{2}\right)^2 + \left(\frac{2s_{k+1} + s + n - 2k}{2}\right)^2 \right] \phi_\sigma\left(\frac{\lambda}{2}\right), \tag{6.2}$$

where

$$\phi_\sigma(z) = \begin{cases} \coth \pi z & \text{if } s + n \text{ is even}, \\ \operatorname{th} \pi z & \text{if } s + n \text{ is odd}. \end{cases}$$

The highest weight of $\sigma'_{p,q}$ was calculated in (3.10), but we used Baldoni–Silva's parametrization of highest weights, which is slightly different from (although equivalent to) Miatello's one. Namely, according to [1] or [2], highest weights of elements $\sigma \in \widehat{M}$ are written in the form

$$\mu_\sigma = \frac{b_1}{2}(\varepsilon_1 + \varepsilon_{n+1}) + \sum_{k=2}^{n} b_k\, \varepsilon_k,$$

with $b_1, \ldots, b_n \in \mathbb{Z}$ and $b_2 \geq \cdots \geq b_n$. Since $\sum_{k=1}^{n+1} \varepsilon_k = 0$, the dictionary is given by $s_k = b_k - b_n$ and $s = 2b_n - b_1$. Hence, for $\sigma = \sigma'_{p,q}$ we get

$$\begin{aligned} &s_2 = \cdots = s_{q+1} = 2, \\ &s_{q+2} = \cdots = s_{n-p} = 1, \\ &s_{n-p+1} = \cdots = s_n = 0, \\ &s = q - p - 2. \end{aligned}$$

An easy calculation leads to

$$\begin{aligned} &\prod_{k=1}^{n-1} \left[\left(\frac{\lambda}{2}\right)^2 + \left(\frac{2s_{k+1} + s + n - 2k}{2}\right)^2 \right] \\ &\quad = 2^{2-2n} \prod_{k=0}^{n} [\lambda^2 + (n - p + q - 2k)^2][\lambda^2 + (n - p - q)^2]^{-1} \\ &\qquad \times [\lambda^2 + (-n + p + q)^2]^{-1}. \end{aligned}$$

Changing the index k by $n - p + q - 2k$, we expand the right-hand side of (6.3) by discussing on the parity of $n - p + q$. Set m and M as in the statement of the theorem, and note that $|n - p - q| \leq m$. Suppose for instance $n - p + q$ even. Then

$$\prod_{k=1}^{n-1}\left[\left(\frac{\lambda}{2}\right)^2+\left(\frac{2s_{k+1}+s+n-2k}{2}\right)^2\right]$$
$$=2^{2-2n}\lambda^2\prod_{\substack{k=2\\ k\text{ even}\\ k\neq=n-p-q}}^{m}(\lambda^2+k^2)^2\prod_{\substack{k=m+2\\ k\text{ even}}}^{M}(\lambda^2+k^2). \tag{6.3}$$

Besides, $\phi_\sigma(\lambda/2)=\coth(\pi\lambda/2)$, and according to (6.2), it remains only to calculate the dimension of $\sigma'_{p,q}$.

Lemma 6.2.
(i) *if* $0\le p+q\le n-1$,

$$\dim\sigma'_{p,q}=\frac{n-p-q}{n}\binom{n}{p}\binom{n}{q}=\frac{n(n-p-q)}{(n-p)(n-q)}\dim\sigma_{p,q};$$

(ii) *if* $0\le p+q\le n$,

$$\dim\tau'_{p,q}=\frac{n+1-p-q}{n+1}\binom{n+1}{p}\binom{n+1}{q}=\frac{(n+1)(n+1-p-q)}{(n+1-p)(n+1-q)}\dim\tau_{p,q}.$$

Assertion (i) follows, after a long calculation, from Weyl's dimension formula (see e.g. [24, Theorem 5.84]). Assertion (ii), (which is useless here, but given for the sake of completeness) can be proved in the same manner or, more simply, by using the M-decomposition of $\tau'_{p,q}$ (Proposition 3.1).

Putting together (6.2), (6.4) and the lemma, one gets assertion (i) of the theorem. The calculation is similar in the odd case, and we skip the details. As regards the poles of the Plancherel density, it suffices to remind the well-known Eulerian developments:

$$\pi z\coth\pi z=1+\sum_{k=1}^{+\infty}\frac{2z}{z^2+k^2},\qquad \frac{\operatorname{th}\pi z}{\pi z}=\frac{2}{\pi^2}\sum_{k=1}^{+\infty}\frac{1}{z^2+k^2}.$$

Thus the theorem is proved. □

Remark. *In each result in which the Plancherel measure occured (Sections 3 and 5), the integration was performed on $\mathbb{R}_+$. Therefore, in these formulas, the expressions of the Plancherel densities given in the theorem must be divided by 2.*

Next result, although a priori of little importance, answers a natural question when one deals with principal series representations.

Corollary 6.3. *For any p,q, the principal series representation $\pi_{\sigma'_{p,q},0}$ is irreducible (or, equivalently, has zero corresponding Plancherel density).*

Proof. A necessary and sufficient condition for such a representation to be reducible is recalled in a more general setting in Corollary 14.30 of Knapp [23], namely: if $P=MAN$

is a parabolic subgroup of a semisimple Lie group G with $\dim A = 1$ and $\sigma \in \widehat{M}$, then the corresponding principal series representation $\pi_{\sigma,0}$ is *reducible* if and only if:
(a) the Weyl group $W = W(\mathfrak{g}, \mathfrak{a})$ has order 2,
(b) if w denotes the nontrivial element, $w \cdot \sigma = \sigma$, and
(c) the Plancherel density $\nu_\sigma(\lambda)$ verifies $\nu_\sigma(0) \neq 0$.
For $\sigma = \sigma'_{p,q}$ conditions (a) and (b) hold (see the proof of Theorem 3.2), but condition (c) falls since ν_σ has a (second order) zero at $\lambda = 0$ by the previous theorem. □

7. The heat kernel

Fix $\tau = \tau'_{p,q}$, as usual. For $t > 0$, the *heat kernel* associated with (primitive) differential (p, q)-forms on $H^n(\mathbb{C})$ is the τ-radial function H_t on G such that

$$\mathcal{H}^\tau_{\sigma,\lambda}(H_t) = \mathrm{e}^{-t[\lambda^2+(n-k_\sigma)^2]} \tag{7.1}$$

for each $\sigma \in \widehat{M}(\tau)$ (k_σ was defined in (5.3)). Note that H_t belongs to the *τ-radial Schwartz space* $\mathcal{S}(G, K, \tau, \tau)$. Details about the heat equation in the Schwartz setting and a motivation for definition (7.1) can be found in [33, Section 7].

The expression of H_t is given by the inversion formulas (5.6) and (5.7), which obviously hold in the Schwartz setting by density. As an example, if $0 \leq p + q \leq n - 1$, then

$$H_t(x) = \sum_{\sigma\in\widehat{M}(\tau)} c_\sigma \int_0^{+\infty} \mathrm{d}\nu_\sigma(\lambda)\, \mathrm{e}^{-t[\lambda^2+(n-k_\sigma)^2]}\, \Phi^\tau_{\sigma,\lambda}(x). \tag{7.2}$$

It is clear also that the definition of the heat kernel can be extended to the Schwartz space of $\tau_{p,q}$-radial functions on G by using Proposition 2.1.

Next result gives some information about the decay at infinity of the heat kernel $H_t(x)$ associated with (p, q)-forms on $H^n(\mathbb{C})$, when $x = e$ is the neutral element of G. Recall that the standard notation $a(t) \underset{t\to+\infty}{\sim} b(t)$ means $\lim_{t\to+\infty} a(t)/b(t) = 1$.

Proposition 7.1. *Let $H_t \in \mathcal{S}(G, K, \tau_{p,q}, \tau_{p,q})$ be the heat kernel associated with differential (p, q)-forms on $H^n(\mathbb{C})$. Then:*
(i) *if $p + q \neq n$, $\mathrm{tr}\, H_t(e) \underset{t\to+\infty}{\sim} c\, t^{-3/2}\mathrm{e}^{-t(n-p-q)^2}$, where $c > 0$ is some constant;*
(ii) *if $p + q = n$, $\mathrm{tr}\, H_t(e) \underset{t\to+\infty}{\sim} c'$, where $c' > 0$ is some constant.*

Proof. Let us assume first $H_t \in \mathcal{S}(G, K, \tau'_{p,q}, \tau'_{p,q})$ and $p + q \neq n$. Then (7.2) implies

$$H_t(e) = \sum_{\sigma\in\widehat{M}(\tau)} c_\sigma \mathrm{e}^{-t(n-k_\sigma)^2} \int_0^{+\infty} \mathrm{d}\nu_\sigma(\lambda) \mathrm{e}^{-t\lambda^2}\, \mathrm{Id}_\tau. \tag{7.3}$$

For convenience, set

$$\eta_\sigma(t) := \int_0^{+\infty} \mathrm{d}\nu_\sigma(\lambda)\mathrm{e}^{-t\lambda^2}.$$

Put $\zeta = t^{1/2}\lambda$. We look for an equivalent at infinity for the Plancherel density $\nu_\sigma(t^{-1/2}\zeta)$. Consider for instance the case $n - p + q$ even (the calculation is similar in the odd case). By Theorem 6.1, $\nu_\sigma(t^{-1/2}\zeta) \underset{t\to+\infty}{\sim} \mathrm{cst}\, t^{-1}$, so that

$$\eta_\sigma(t) \underset{t\to+\infty}{\sim} \mathrm{cst} \int_0^{+\infty} \mathrm{d}\zeta\, t^{-3/2}\mathrm{e}^{-\zeta^2}.$$

Hence, by (5.3) and (7.3),

$$\mathrm{tr} H_t(e) \underset{t\to+\infty}{\sim} \mathrm{cst}\, t^{-3/2}\mathrm{e}^{-t(n-p-q)^2}.$$

Now, if $H_t \in \mathcal{S}(G, K, \tau_{p,q}, \tau_{p,q})$, we know (Corollary 4.3) that the Laplacian spectrum is entirely determined by its restriction to primitive (p, q)-forms. Thus our estimate still holds and this proves assertion (i) of the proposition.

On the other hand, if $p + q = n$, assertion (ii) is clear since in this case the dominant term is the discrete one (see (5.7)). □

Remark.

1. *An immediate consequence of Proposition* 7.1 *is the calculation of the rth Novikov–Shubin invariant* $\alpha_r(M)$ *(see e.g.* [27, *Section* 5] *for a general definition) of any locally symmetric space M whose universal covering is* $H^n(\mathbb{C})$. *Namely we have*

$$\alpha_r(M) = \begin{cases} +\infty & \text{if } r \neq \frac{1}{2}\dim M \\ 0 & \text{if } r = \frac{1}{2}\dim M \end{cases}$$

   and recover thus in our particular case the observation made in [27, *Section* 7] *about the Novikov–Shubin invariants of Kähler hyperbolic manifolds.*
2. *Haar measures given in Section* 6 *are settled in order to recover the classical result*

$$\mathrm{tr}\, H_t(e) \underset{t\to0^+}{\sim} (4\pi t)^{-n}$$

   (see [28, *Section* 3]).

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